Diffeomorphism of manifold is equivalent

example let $(M, \{(U_\alpha, \phi_\alpha) : \alpha \in I_1\})$

and $(N, \{(V_\beta, \psi_\beta) : \beta \in I_2\})$ be smooth man

of dimension ~~M and N~~ m and n respectively

Then $M \times N$ is also a smooth manifold of dimen

$(m+n)$ with smooth structure given by

$$\{(U_\alpha \times V_\beta, \phi_\alpha \times \phi_\beta) : \alpha \in I_1 \text{ and } \beta \in I_2\}$$

where maps

$\phi_\alpha \times \phi_\beta : U_\alpha \times V_\beta \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$ is given by

$$(\phi_\alpha \times \phi_\beta)(p, q) = (\phi_\alpha(p), \phi_\beta(q))$$

Lie group: let G be a group which have also a smooth structure with respect to product manifold on $G \times G$

If the multiplication map

$$G \times G \longrightarrow G$$
$$(x, y) \longrightarrow x \cdot y$$

and the inverse map $G \longrightarrow G$ given by $x \longrightarrow x^{-1}$ are smooth then G is called Lie group.

ample :- consider $\mathbb{R}$ which is group with usual addition of real number Also $\mathbb{R}$ is manifold with ~~usual~~ & ~~usal~~ usual smooth structure

consider the map

$\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ given by

$(x,y) \longrightarrow x+y$

which is smooth and inverse map $\mathbb{R}\longrightarrow\mathbb{R}$ given by $x \longrightarrow -x$ is also smooth

So $\mathbb{R}$ is lie group

$\mathbb{R}^n$ is lie group

R^* is also lie group

S^1 is " " "

$GL(n,\mathbb{R})$ is also Lie group.

ex:- $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$

$(\mathbb{R}^*, \cdot)$ is a group, which is also a 1 smooth manifold with usual smooth structure

consider the map

$\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ given by

$(x,y) \longrightarrow x\cdot y$

which is smooth and inverse map $\mathbb{R}\longrightarrow\mathbb{R}$ given by $x \longrightarrow 1/x$ is also smooth

So $\mathbb{R}^*$ is lie group.

Ex.- $GL(n,\mathbb{R}) = \{A : A^{-1} \text{ exist}\}$

$GL(n,\mathbb{R})$ is a group which is also smooth manifo[ld] with smooth structure

consider the map

$$GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$$
$$(A,B) \longrightarrow A\cdot B \quad \text{which is smooth}$$

and inverse map $GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ Given by

$$A \longrightarrow A^{-1} \quad \text{which is smoo[th]}$$

So $GL(n,\mathbb{R})$ is lie group

Tangent vector on smooth manifold :- let M be a smooth manifold and $p \in M$. A tangent vector at $p \in M$ is a map

$$X_p : C^{\infty}(p) \longrightarrow \mathbb{R} \quad \text{such that}$$

(1) $X_p(af + bg) = aX_p(f) + bX_p(g)$ (linearly property)

(2) $X_p(f\cdot g) = g(p)X_p(f) + f(p)X_p(g)$ $\forall f, g \in C^{\infty}(p)$, $a, b \in \mathbb{R}$

Liebnitz type rule of differentiation

where $C^{\infty}(p)$ denote the set of real valued smooth maps. defined on nbd of p

Example:- let M be smooth manifold and $p \in M$

Define $O_p : C^{\infty}(p) \longrightarrow \mathbb{R}$ by

$O_p(f) = 0 \quad \forall f \in C^{\infty}(p)$

Solⁿ claim O_p is tangent vector at p is called zero vector at p

(1) $O_p(af + bg) = 0 = a\,O_p(f) + b\,O_p(g) \qquad \forall f, g \in C^{\infty}(p)$

(2) $O_p(f \cdot g) = f(p)\,O_p(g) + g(p)\,O_p(f) = 0 \qquad a, b \in \mathbb{R}$

Thus O_p is tangent vector at p

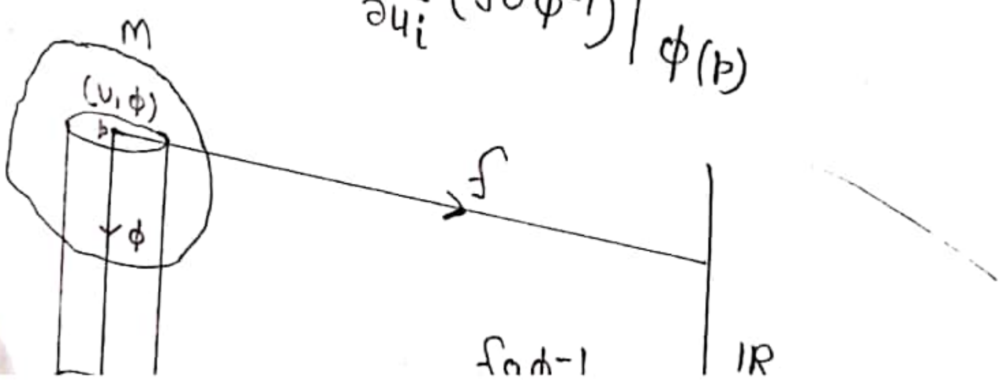
** let M be smooth manifold of dimension n and $p \in M$ let (U, ϕ) be local chart on M around p and x^i be corresponding local coordinate function i.e

$x^i = u_i \circ \phi \qquad 1 \leq i \leq n$

Define

$\frac{\partial}{\partial x^i}\Big|_p : C^{\infty}(p) \longrightarrow \mathbb{R}$ as $\quad 1 \leq i \leq n$

$\frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial}{\partial u_i}(f \circ \phi^{-1})\Big|_{\phi(p)}$



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Claim $\frac{\partial}{\partial x^i}\Big|_p$ is tangent vector at $p \in M$ $1 \le i \le n$

(1) let $a, b \in \mathbb{R}$ and $f, g \in C^\infty(p)$

$$\frac{\partial}{\partial x^i}\Big|_p (af+bg) = \frac{\partial}{\partial u_i}\left((af+bg)\circ\phi^{-1}\right)\Big|_{\phi(p)}$$

$$= \frac{\partial}{\partial u_i}\left(af\circ\phi^{-1} + bg\circ\phi^{-1}\right)\Big|_{\phi(p)}$$

$$= a\frac{\partial}{\partial u_i}(f\circ\phi^{-1})\Big|_{\phi(p)} + b\frac{\partial}{\partial u_i}(g\circ\phi^{-1})\Big|_{\phi(p)}$$

$$\frac{\partial}{\partial x^i}\Big|_p (af+bg) = a\frac{\partial}{\partial x^i}\Big|_p (f) + b\frac{\partial}{\partial x^i}\Big|_p (g)$$

(2) $$\frac{\partial}{\partial x^i}\Big|_p (fg) = \frac{\partial}{\partial u_i}(fg\circ\phi^{-1})\Big|_{\phi(p)}$$

$$= \frac{\partial}{\partial u_i}\left((f\circ\phi^{-1})\cdot(g\circ\phi^{-1})\right)\Big|_{\phi(p)}$$

$$= (f\circ\phi^{-1})_{\phi(p)}\frac{\partial}{\partial u_i}(g\circ\phi^{-1})\Big|_{\phi(p)}$$

$$+ (g\circ\phi^{-1})_{\phi(p)}\frac{\partial}{\partial u_i}(f\circ\phi^{-1})\Big|_{\phi(p)}$$

$$= f(p)\frac{\partial}{\partial x^i}\Big|_p (g) + g(p)\frac{\partial}{\partial x^i}\Big|_p (f)$$

Thus $\frac{\partial}{\partial x_i}\Big|_p$ $i = 1, 2, \ldots\ldots n$ are tangent vector at p

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ex: $\mathbb{R}^3$

$$\frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p \frac{\partial}{\partial z}\Big|_p$$

$$X_p(f) = X_p \cdot (\nabla f)_p$$

$$i_p = (1,0,0)_p$$

$$i_p(f) = i_p \cdot (\nabla f)_p$$

$$= (1,0,0)_p \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)(p)$$

$$= \frac{\partial f}{\partial x}\Big|_p$$

$$= \frac{\partial}{\partial x}\Big|_p (f)$$

$$i_p = \frac{\partial}{\partial x}\Big|_p$$

(*) let T_pM denote the set of tangent vector at p

Define $(X_p+Y_p)(f) = X_p(f) + Y_p(f)$

$$c(X_p)(f) = cX_p(f)$$

Then the set T_pM denote a vector space over $\mathbb{R}$ called the tangent space T_pM at $p \in M$

Result:- $X_p(c) = 0$ where c is fixed real number

$$X_p(1) = X_p(1\cdot 1)$$

$$= 1X_p(1) + 1\cdot X_p(1)$$

$$\Rightarrow X_p(1) = 0$$

Now $X_p(c) = X_p(c\cdot 1) = cX_p(1) = c\cdot 0 = 0$

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Path in manifold :- let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$

be a smooth curve in M passing through $\gamma(0)$ say p

Define

$$\gamma'(0)(f) = \frac{d}{dt}(f \circ \gamma)(0) \qquad \forall f \in C^{\infty}(p)$$

claim $\gamma'(0)$ is tangent vector.

(1) let $a, b \in \mathbb{R}$ and $f, g \in C^{\infty}(p)$

$$\gamma'(0)(af + bg) = \frac{d}{dt}(af + bg) \circ \gamma$$

$$= \frac{d}{dt}(a f \circ \gamma + b g \circ \gamma)$$

$$= a \frac{d}{dt}(f \circ \gamma)\Big|_0 + b \frac{d}{dt}(g \circ \gamma)\Big|_0$$

$$= a\gamma'(0)f + b\gamma'(0)g$$

(2) $$\gamma'(0)(fg) = \frac{d}{dt}(fg \circ \gamma)\Big|_0$$

$$= \frac{d}{dt}\left[(f \circ \gamma)(g \circ \gamma)\right]\Big|_0$$

$$= (f \circ \gamma)(0) \frac{d}{dt}(g \circ \gamma)(0) + (g \circ \gamma)(0) \frac{d}{dt}(f \circ \gamma)$$

$$= \cancel{\gamma'(0)(f)}$$

$$= f(p)\, \gamma'(0)(g) + g(p)\, \gamma'(0)(f)$$

#

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$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at x

$Df(x): \mathbb{R}^n \longrightarrow \mathbb{R}$ be linear map given by

$$Df(x)(v) = \langle \operatorname{grad} f(x), v \rangle$$

where $\langle \ \rangle$ is Euclidean inner product.

<u>Taylors formula</u> :- let $F: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth maps defined in nbd of point $x = 0$

Let $\phi: [0,1] \longrightarrow \mathbb{R}^n$ be a map defind as $\phi(t) = tx$ for some x in nbd of zero.

Then

$F \circ \phi : [0,1] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$(F\circ\phi)(1) = (F\circ\phi)(0) + \int_0^1 (F\circ\phi)'(t)\,dt \quad \text{—— (1)}$$

By chain rule of differentiability

$$(F\circ\phi)'(t) = DF(\phi(t))\cdot D\phi(t)$$

$$= DF(tx)\cdot x$$

$$= \langle \operatorname{grad} F(tx), x \rangle$$

$$= \sum_{i=1}^{n} x_i \frac{\partial F}{\partial u_i}(tx)$$

From eqn (1) becomes

$$F(x) = F(0) + \sum_{i=1}^{n} \int_0^1 x_i \frac{\partial F}{\partial u_i}(tx)\,dt$$

$$F(x) = F(0) + \sum_{i=1}^{n} x_i \, g_i(x)$$

where $g_i(x) = \int_0^1 \frac{\partial F}{\partial u_i}(tx)\,dt$

$$g_i(0) = \int_0^1 \frac{\partial F}{\partial u_i}(0)\,dt$$

$$= \frac{\partial F}{\partial u_i}(0)$$

$$= \frac{\partial}{\partial u_i}\Big|_0 (F)$$

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Theorem: let M be a n dimensional smooth manifold let $P \in M$ than

$$\dim(T_pM) = \dim M = n$$

where T_pM denote the tangent space to M at P

Proof Let (u, ϕ) be the local chart on M centred at P

i.e $\phi(p) = 0$

let x^i be the corresponding local coordinate for $1 \le i \le n$

Then $\frac{\partial}{\partial x^i}\Big|_p$ $1 \le i \le n$ are tangent vector at P

consider the set

$$B = \left\{ \frac{\partial}{\partial x^i}\Big|_p \quad 1 \le i \le n \right\} \subseteq T_pM$$ we shall show that

B is basis of T_pM

To show that B is l.i consider

$$\sum a^i \frac{\partial}{\partial x^i}\Big|_p = 0 \text{ (vector)} \qquad a_i \in \mathbb{R}$$

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$$\Rightarrow \sum a^i \frac{\partial}{\partial x^i}\Big|_p (x^j) = 0(x^j) \qquad 1 \le j \le n$$

$$\Rightarrow \sum a^i \frac{\partial}{\partial u_i} (x^j \circ \phi^{-1}) \Big| \phi(p) = 0$$

$$\Rightarrow \sum a^i \frac{\partial}{\partial u_i} ((u_j \circ \phi) \circ \phi^{-1}) \Big| \phi(p) = 0$$

$$\sum a^i \frac{\partial}{\partial u_i} (u_j) \Big| \phi(p) = 0$$

$$\sum a^i \delta_i^j = 0$$

$$\Rightarrow a^j = 0 \qquad 1 \le j \le n$$

So B is L.I

now we shall show that B generate T_pM i.e

$$T_pM = [B]$$

let $X_p \in T_pM$ then we shall show that

$$X_p = \sum X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$$

we shall transform the problem to one on the open set $\phi(U)$ of $\mathbb{R}^n$ let $f \in C^\infty(P)$ and define $F = f \circ \phi^{-1}$ then

$F \in C^\infty(0)$ in $\mathbb{R}$

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}$$

The first order Taylor expansion for F is

$$F(x) = F(0) + \sum_{i=1}^{n} x^i G_i(x)$$

where G_i are smooth function arround 0 and

$$G_i(0) = \frac{\partial}{\partial u_i}(F(0)) \qquad (1)$$

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we now pull this expansion back to u via ϕ and obtain

$$f(x) = f(P) + \sum_i x^i g_i(x) \quad \text{——— (2)}$$

where $g_i = G_i \circ \phi$ and $x^i = u_i \circ \phi$

now applying X_P to g_i equation (2) we get

$$X_P(f) = \overset{0}{\overset{\parallel}{X_P(f(P))}} + \sum X_P(x^i) g_i(P) + \overset{0}{\overset{\parallel}{x^i(P)}} X_P(g_i)$$

$\{ \because x^i(P) = 0$ as $\phi(P) = 0$

$$X_P(f) = \sum_i X_P(x^i) g_i(P) \quad \text{——— (3)}$$

But we have

$$\begin{aligned} g_i(P) &= (G_i \circ \phi)(P) \\ &= G_i(\phi(P)) \\ &= G_i(0) = \frac{\partial}{\partial u_i} F(0) \\ &= \frac{\partial}{\partial u_i}\Big|_{\phi(P)} (f \circ \phi^{-1}) \\ &= \frac{\partial}{\partial x^i}\Big|_P (f) \end{aligned}$$

then equation (3) becomes

$$X_P(f) = \sum_i X_P(x^i) \frac{\partial}{\partial x^i}\Big|_P (f)$$

$$\boxed{X_p = \sum_{i=1}^{n} X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p}$$

thus B generate T_pM and hence it is basis of T_pM

therefore

$$\dim T_pM = \dim M = n.$$

* A map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be differentiable at a point $x \in \mathbb{R}^n$ if there exist continuous linear map

$A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

such that $f(x+h) - f(x) = A(h) + o(\|h\|)$ holds for all $h \in \mathbb{R}^n$

or equivalently

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A(h)\|}{\|h\|} = 0$$

linear map A is called derivative of f at x and is denoted as $f'(x)$ or $Df(x) = A$

ex:- let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous linear map

what is $Df(x)$ $x \in \mathbb{R}^n$

Sol^n $f(x+h) = f(x) + f(h)$

$f(x+h) - f(x) = f(h)$

we take $A(h) = f(h)$

i.e $Df(x) = A = f$

$(Df(x))(h) = A(h) = f(h)$

ex:- let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be given as

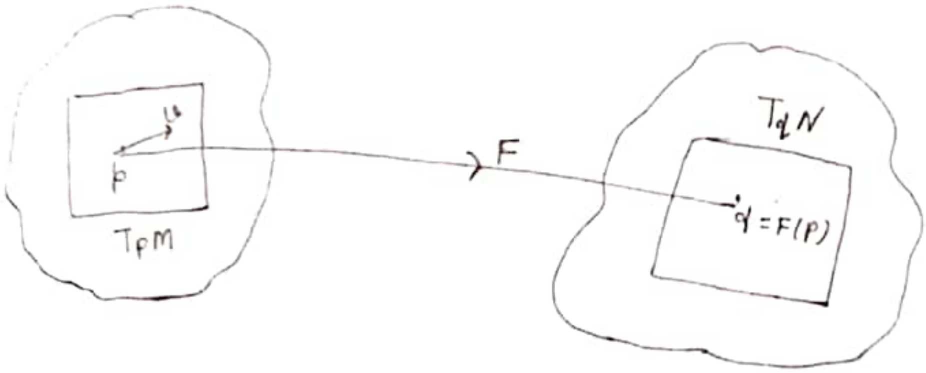
$$f(x) = Ax + v$$

which $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map and v is fix vector what is $Df(x)$

$$f(x+h) - f(x) = [A(x+h)+v] - [Ax+v]$$
$$= \cancel{Ax} + Ah + \cancel{v} - \cancel{Ax} - \cancel{v}$$
$$= Ah$$

$$Df(x) = A$$

Derivative of smooth map - let M and N be smooth manifo of dimension m and n resp. at $F: M \longrightarrow N$ is a smooth map let $p \in M$ and $q = F(p)$ then derivative



of F at p denoted by $DF(p)$ is a linear map

$$DF(p): T_pM \longrightarrow T_qN \text{ given as}$$

$$DF(p)(v)(g) = v(g \circ F) \quad \text{where } g \in C^\infty(q)$$
$$v \in T_pM$$

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claim: $Df(p) : T_pM \longrightarrow T_qN$ is a linear transformation

$$Df(p)(a_1v_1 + a_2v_2)(g) = (a_1v_1 + a_2v_2)(g\circ f)$$

$$= a_1v_1(g\circ f) + a_2v_2(g\circ f)$$

$$= a_1 Df(p)(v_1)(g) + a_2 Df(p)(v_2)(g)$$

$$\forall\, g \in C^\infty(q)$$

matrix of $Df(p)$

$Df(p) : T_pM \longrightarrow T_qN \qquad q = f(p)$

$Df(p)(v)(g) = v(g\circ f) \quad \forall\, v \in T_pM,\ g \in C^\infty(q)$

let (U,ϕ) be a local coordinate system around p and (V,ψ) be that along $q = f(p)$ with local coordinates x^i and y^α respectively $1 \le i \le m$ and $1 \le \alpha \le n$

then standard basis of T_pM and T_qN are

$\left\{ \frac{\partial}{\partial x^i}\Big|_p \ 1 \le i \le m \right\}$ and $\left\{ \frac{\partial}{\partial y^\alpha}\Big|_q \ 1 \le \alpha \le n \right\}$

$Df(p) : T_pM \longrightarrow T_qN$

let $v \in T_pM$ and $w = Df(p)(v) \in T_qN$

so $w = \sum_{\alpha=1}^{n} w(y^\alpha) \frac{\partial}{\partial y^\alpha}\Big|_q$

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$$w = \sum_{j=1}^{n} DF(p)(v)(y^j) \frac{\partial}{\partial y^j}\Big|_q$$

$$= \sum_{j=1}^{n} v(y^j \circ F) \frac{\partial}{\partial y^j}\Big|_q$$

now as $v \in T_pM$

we can take $v = \frac{\partial}{\partial x^i}\Big|_p$

Then $w = \sum_{j=1}^{n} \frac{\partial}{\partial x^i}\Big|_p (y^j \circ F) \frac{\partial}{\partial y^j}\Big|_q$

$$DF(p)\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x^i}\Big|_p (y^j \circ F) \frac{\partial}{\partial y^j}\Big|_q$$

So matrix representation DF(p) with local coordinate system

$$\left(\frac{\partial}{\partial x^i}\Big|_p (y^j \circ F)\right)_{\substack{1\le i\le m \\ 1\le j\le n}}$$ which is called jacobian matrix

of F at p relative to local co-ordinate system.

Vector fields:- let M be smooth manifold. A map

$X : p \longrightarrow X_p \in T_pM \quad \forall\, p \in M$ is called a vector field on M

OR

let $C^\infty(M)$ denote the set of real valued smooth maps on M then a vector field X on M is a map

$$X : C^\infty(M) \longrightarrow C^\infty(M) \quad \text{s.t}$$

① $X(af+bg) = a(Xf) + b(Xg)$.

(2) $X(fg) = g(Xf) + f(Xg) \quad \forall\, a,b \in \mathbb{R} \ \forall\, f,g \in C^\infty(M)$

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let (U, ϕ) be a local coordinate system around p with local coordinate (x^i)

then $X_p = \sum_{i=} X_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$

Assigenment $p \longrightarrow X_p \in T_pM$ is said to be smooth if $X_p(x^i)$ are smooth function w.r.t. local chart (U, ϕ).

ex:① let $M = \mathbb{R}^n$

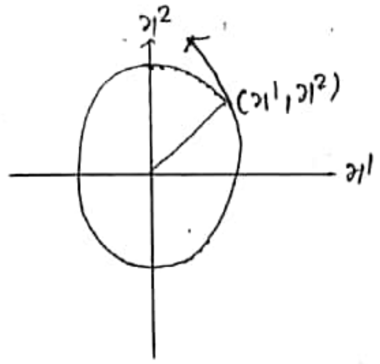
and let $X_i = \frac{\partial}{\partial u_i} : p \longrightarrow \frac{\partial}{\partial u_i}\Big|_p \in T_pM$

so $\frac{\partial}{\partial u_i}$ $(1 \le i \le n)$ are vector field on $\mathbb{R}^n$ which are also called base vector field

If X is any vector field of $\mathbb{R}^n$ then

$X = \sum f_i X_i$ where $f_i \in C^\infty(M)$

ex:2 let $M = S^1 = \{ \underset{\in \mathbb{R}^2}{(x^1, x^2)} : (x^1)^2 + (x^2)^2 = 1 \}$



$x^1 = \cos t$

$x^2 = \sin t$

$\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = (-\sin t, \cos t)$

$= (-x^2, x^1)$

let $p = (x^1, x^2) \in S^1$

then $X_p = -x^2 \frac{\partial}{\partial x^1}\Big|_p + x^1 \frac{\partial}{\partial x^2}\Big|_p \in T_pS^1$

$X = \left(-x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}\right)$ is a vector field on S^1

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